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THE PERSOZ'S GEPHYROIDAL MODEL DESCRIBED BY A MAXIMAL MONOTONE DIFFERENTIAL INCLUSION

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Abstract. *The rheological Persoz's gephyroidal model, made out of some elementary rheological models (dry friction element and linear spring) can be covered by the existence and uniqueness theory for maximal monotone operators. Moreover, classical results of numerical analysis allow to use a numerical implicit Euler scheme, with order of convergence one. Some numerical simulations are presented.*

1 Introduction

In previous works [1, 2, 3, 4, 5, 6] dynamical behaviours of mechanical systems involving friction have been studied. Associations of springs, dashpots, Saint-Venant (also called Maxwell elements) in parallel or in series have been investigated. In the book [7] Persoz introduced similar nonlinear models for quasi-static behaviours. He denoted models M (for Maxwell) or K (for Kelvin). Finally, he distinguished two classes of associations of such elements:

- the first one (involving M or K) can be modelled by parallel or series associations: It has been called mixed models,
- the second one has been called general models or gephyroid models.

Among these latter ones, Persoz proposed a very simple –especially significant and clearly non-mixed – gephyroid model of quasi-static behaviour.

This work is based on [8]. Here we examine this model in general. The paper is organised as follows: in Section 2, model is described. In Section 3, numerical scheme is presented based on mathematical results. In Section 4, we mainly propose numerical example of dynamical behaviour.

2 Description of the model

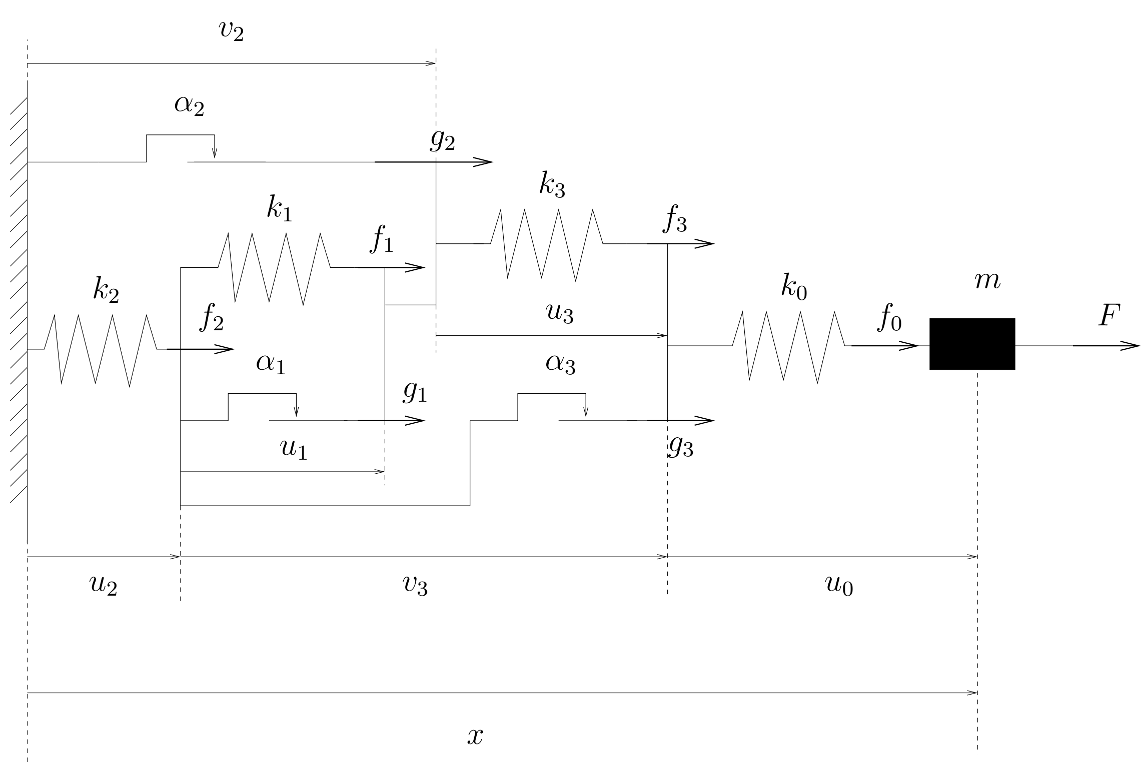

Figure 1: The studied model with forces f_i and g_i and displacement u_i, v_i and x.

We introduce the model of figure 1. The notation are analogous to those of [1] :

- For all $i \in \{0, ..., 3\}$, the displacement of spring with stiffness k_i is denoted by u_i and the force exerted by this spring is denoted by f_i ;

- For all $i \in \{2,3\}$, the displacement of St-Venant elements with threshold α_i is denoted by v_i and the force exerted by this element is denoted by g_i ;
- Since the spring with stiffness k_1 and the St-Venant element with threshold α_1 are connected in parallel, we do not introduce the displacement v_1, equal to u_1 and we denote by g_1 the force exerted by this St-Venant element;
- Let x be the abscissa of material point with mass m, and F be the external force, applied to this point.

We consider σ the multivalued graph sign defined by

$$\sigma(x) = \begin{cases} -1 & \text{if } x < 0, \\ 1 & \text{if } x > 0, \\ [-1,1] & \text{if } x = 0. \end{cases} \tag{1}$$

The reader is refered to [9] for notions of multivalued operator. Following [1], the different equations governing the model are given below. First, the geometrical connexion is written as:

$$u_0 + v_3 + u_2 = x, \tag{2a}$$
$$v_2 + u_3 + u_0 = x, \tag{2b}$$
$$u_1 + u_3 = v_3. \tag{2c}$$

The constitutive laws of springs and St-Venant elements are:

$$\forall i \in \{0, ..., 3\}, \quad f_i = -k_i u_i, \tag{2d}$$
$$\forall i \in \{2,3\}, \quad g_i \in -\alpha_i \sigma\left(\dot{v}_i\right), \tag{2e}$$
$$g_1 \in -\alpha_1 \sigma\left(\dot{u}_1\right). \tag{2f}$$

The equilibrium leads to

$$g_2 + f_1 + g_1 = f_3, \tag{2g}$$
$$g_3 + f_3 = f_0, \tag{2h}$$
$$g_3 + g_1 + f_1 = f_2, \tag{2i}$$
$$m\ddot{x} = f_0 + F. \tag{2j}$$

Let us introduce β the inverse graph of σ, defined by

$$\beta(x) = \begin{cases} \emptyset & \text{if } x \in]-\infty,-1[\cup]1,+\infty[, \\ \{0\} & \text{if } x \in]-1,1[, \\ \mathbb{R}_- & \text{if } x = -1, \\ \mathbb{R}_+ & \text{if } x = 1. \end{cases} \tag{3}$$

We consider the convexe C of $\mathbb{R}^3$ defined by

$$C = [-\alpha_2, \alpha_2] \times [-\alpha_3, \alpha_3] \times [-\alpha_1, \alpha_1], \tag{4}$$

and we consider the multivalued operator from $\mathbb{R}^3$ to $\mathbb{R}^3$ defined by

$$\forall W = (w_1, w_2, w_3) \in \mathbb{R}^3, \quad A(W) = \beta\left(\frac{W_1}{\alpha_2}\right) \times \beta\left(\frac{W_2}{\alpha_3}\right) \times \beta\left(\frac{W_3}{\alpha_1}\right). \tag{5}$$

By elimination of some unknows, we prove that (2) is equivalent to the system

$$\left\{\begin{array}{l} \dot{x} = y, \\ \dot{y} = \frac{1}{m}\left(F - \delta x + EW\right), \\ \dot{W} + KA(W) \ni -k_0 U y, \end{array}\right. \tag{6}$$

where K is the matrix defined by

$$K = \begin{pmatrix} k_0 + k_2 & k_0 & -(k_0 + k_2) \\ k_0 & k_0 + k_3 & -(k_0 + k_3) \\ -(k_0 + k_2) & -(k_0 + k_3) & k_0 + k_1 + k_2 + k_3 \end{pmatrix}, \tag{7}$$

and E and W are given by

$$U = \begin{pmatrix} 1, \\ 1, \\ -1 \end{pmatrix}, \quad W = \begin{pmatrix} g_2 \\ g_3 \\ g_1 \end{pmatrix}, \quad E = k_0 U^T K^{-1}. \tag{8}$$

System (6) can be written under the form

$$\left\{\begin{array}{l} \dot{X}(t) + M\mathcal{A}(X(t)) \ni G\big(t, X(t)\big), \text{a.e. on }]0, T[, \\ X(0) = X_0 = (x_0, \dot{x}_0, w_{1,0}, w_{2,0}, w_{3,0}), \end{array}\right. \tag{9}$$

where M is a matrix, G a function from $[0, T] \times \mathbb{R}^5$ to $\mathbb{R}^5$, and $\mathcal{A}$ is a multivalued operator from $\mathbb{R}^5$ to $\mathbb{R}^5$.

3 Existence and uniqueness results and numerical scheme

The matrix K defined by (7) is symmetric positive definite if and only if the numbers $(k_i)_{0 \leq i \leq 3}$ satisfy the following assumption

$$k_0 = 0, \quad \forall i \in \{1, 2, 3\}, \quad k_i > 0, \tag{10a}$$

or

$k_0 > 0$ and at least two numbers among k_1, k_2 and k_3 are non negative (10b)

and then, according to results proved in [1, 4, 3], the solution (6) (or of (9)) exists and is unique.

According to results proved in [1, 3, 4], we considere the numerical scheme

$$x^{p+1} = hy^p + x^p, \tag{11a}$$

$$y^{p+1} = \frac{h}{m}\left(F(t_p) - \delta x^p + EW^p\right) + y^p, \tag{11b}$$

$$W^{p+1} = \text{proj}_{C,K^{-1}}\left(W^p - hk_0 y^p U\right), \tag{11c}$$

where $\text{proj}_{C,K^{-1}}$ is the orthogonal projection on the convex C defined by (4) for the norm on $\mathbb{R}^3$ defined by

$$\forall W \in \mathbb{R}^3, \quad \|W\|_{K^{-1}} = \sqrt{W^T K^{-1} W}. \tag{12}$$

This numerical scheme converges to the solution of (6), with an error in $\mathcal{O}(h)$.

4 Applications

4.1 Quasistatic problems

In the quasistatic case, the mass m can be equal to zero. This case can be treated by the proposed method (existence, uniqueness, numerical scheme). The difference is that the problem is expressed in $\mathbb{R}^4$ instead in $\mathbb{R}^5$, as (6).

4.2 Numerical simulations for dynamical case

Parameters α_i and k_i and initial conditions are defined by

$$\forall i \in \{1,2,3\}, \quad \alpha_i = i, \tag{13a}$$
$$\forall i \in \{0,1,2,3\}, \quad k_i = 1, \tag{13b}$$
$$\forall i \in \{1,2,3\}, \quad g_{i,0} = 0, \tag{13c}$$

and we choose

$$x_0 = 0, \quad \dot{x}_0 = 0, \tag{14a}$$
$$m = 1, \tag{14b}$$
$$T = 80, \quad h = 10^{-3}. \tag{14c}$$

The imposed force is defined by

$$F(t) = 200 \sin(6t). \tag{14d}$$

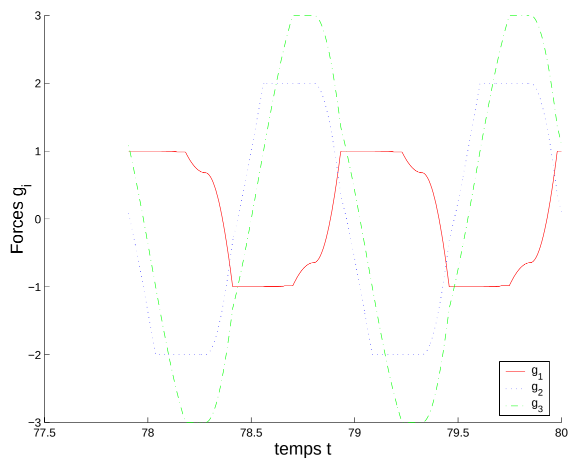

Figure 2: The graphs $(t, g_j(t))$ for $t \in [t_i, t_f]$ and $j \in \{1,2,3\}$.

We plot the graphs $(t, g_j(t))$ for $t \in [t_i, t_f]$ and $j \in \{1,2,3\}$ in Fig. 2. We can observe the specificity of the gephyroidal studied model. Indeed, in Fig. 2 we can see opposite behaviours of functions g_1, g_2 vs g_3. As g_2 and g_3 reach their maximum α_2 and α_3, then g_1 leaves its minimum $-\alpha_1$. Reciprocally, on other intervals, as g_2 and g_3 reach their minimum $-\alpha_2$ and $-\alpha_3$, then g_1 leaves its maximum α_1. From a mechanical point of view, is means that there exist some intervals where dry friction elements 2 and 3 slip in a direction, whereas the dry friction element 1 sticks still in the opposite direction.

5 Conclusion

In this paper, we investigated mainly a simple example of gephyroid model. Wee can see that the dynamical behaviour of such a model is different from a classical mixed model since displacements can exhibit a non classical behaviour (traction when solicitation corresponds to compression e.g.). Mathematical, numerical and main physical properties have been presented; the latter ones are coherent with [7].

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