The Persoz's gephyroidal model described by a maximal monotone differential inclusion

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Schedule

Schedule

Introduction

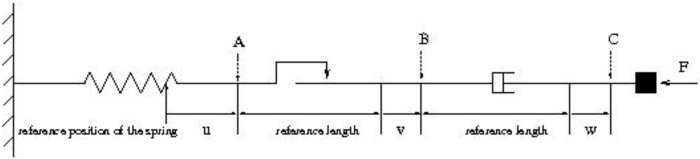
- Applications:
  - ♣ soil-structure coupling
  - ♣ dynamical devices, elastomer blocks (cars, trucks)
  - ♣ etc.
- Modelling
  - ♣ using non smooth elements: dry friction
- Identification
  - ♣ theoretically
  - ♣ from discrete experimental data

Schedule

Assembling simple elements

Simple elements: springs, dashpots, dry friction elements (St-Venant elements).

Classical constitutive laws: example



Mass m, external force F

f: force exerted between two boundaries of simple elements

$$\begin{cases} x = u + v + w \\ f = ku \qquad f \in -\alpha\sigma(\dot{v}) \qquad f = -c\dot{w} \\ \sigma \quad \text{graph of sign function: } \sigma(0) = [-1, 1] \end{cases}$$

Model:

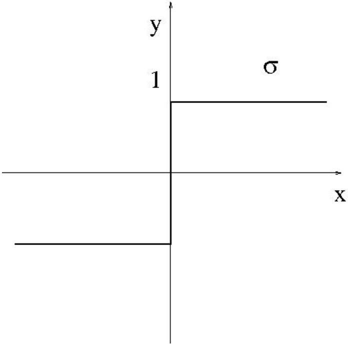
$$\begin{cases} m\ddot{x} = F - ku \\ ku \in \alpha\sigma(\dot{x} - \dot{u} - ku/c) \\ x(0) = x_0, \dot{x}(0) = \dot{x}_0, u(0) = u_0. \end{cases}$$

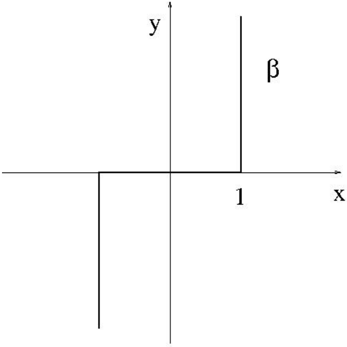
Finally:

$$\begin{cases} \dot{x} = y, \dot{y} = (F - ku)/m, \\ \eta = \alpha/k, \dot{u} + \beta(u/\eta) \ni y - ku/c, \\ x(0) = x_0, \dot{x}(0) = \dot{x}_0, u(0) = u_0 \in [-\eta, \eta]. \\ \beta(z) = \begin{cases} \emptyset \text{ if } z \in]-\infty, -1[\cup]1, +\infty[, \\ \{0\} \text{ if } z \in]-1, 1[, \\ ]-\infty, 0] \text{ if } z = -1, \\ [0, +\infty[\text{ if } z = 1, \end{cases} \end{cases}$$

σ and β : inverse graphs.

$$\sigma = \beta^{-1}.$$

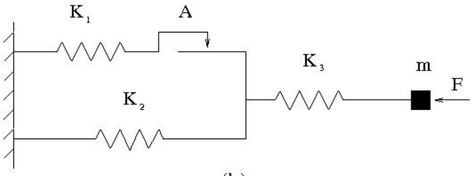




According to Brézis 1973, these graphs (or operators) are maximal monotone (generalized non decreasing functions).

$$\begin{cases} \text{Monotone:} \quad (\sigma(z_1) - \sigma(z_2))(z_1 - z_2) \geq 0, \\ \text{Maximal in the sense of monotone graphs...} \end{cases}$$

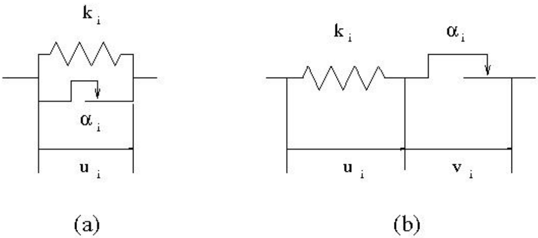
Different combinations ...more or less canonical ...



(b)

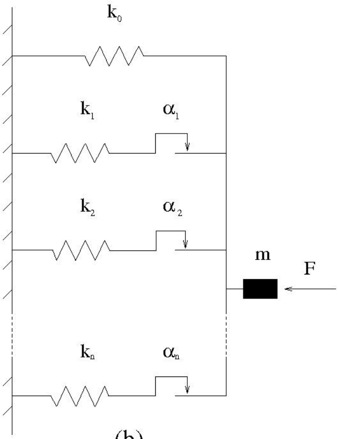
$$
\begin{cases}
\dot{x} = y, \dot{y} = (F - k_0 x - ku)/m, \\
\eta = \alpha/k, \dot{u} + \beta(u/\eta) \ni y, \\
x(0) = x_0, y(0) = \dot{x}_0, u(0) = u_0 \in [-\eta, \eta]. \\
k_0 = \dfrac{K_2 K_3}{K_2 + K_3}, k = \dfrac{K_1 K_3^2}{(K_1 + K_2 + K_3)(K_2 + K_3)}, \\
\alpha = A \dfrac{K_3}{K_2 + K_3},
\end{cases}
$$

We study many combinations built with P_i (parallel) or S_i (series) elements:



...with 1 degree of freedom or n degrees of freedom.

e.g. S_i elements in parallel ...



(b)

Generalized Prandtl model with n S_i elements and one spring...

Mathematical frame

All the previous combinations have the same kind of models.

$$\begin{cases} \forall t \in [0,T], \forall X_1, X_2 \ in \mathbb{R}^p, \|G(t,X_1) - G(t,X_2)\| \leq \omega \|X_1 - X_2\| \\ \forall Y \in \mathbb{R}^p, G(.,Y) \in L^\infty(0,T,\mathbb{R}^p), \\ M \quad \text{symmetric, positive definite,} \\ \phi \quad \text{is convex proper and lower semicontinuous,} \\ \begin{cases} \dot{X}(t) + M\partial\phi(X(t)) \ni G(t,X(t)) \quad \text{a.e. on }]0,T[, \\ X(0) = \xi. \end{cases} \end{cases}$$

$\partial\phi$ subdifferential of ϕ. If $< .,. >$ scalar product in $\mathbb{R}^p$,

$$z \in \partial\phi(Z) \Longleftrightarrow \forall h \in \mathbb{R}^p, \phi(Z+h) - \phi(Z) \geq < z, h > .$$

$$\sigma = \partial|.|, \quad \beta = \partial\psi_{[-1,1]},$$

♣ Existence and uniqueness - Example of result:

Let us give G, ω verifying previous assumptions.

Proposition:

For all $\xi \in D(\partial\phi)$, there exists a unique function $X \in W^{1,1}(0, T, \mathbb{R}^p)$ such that

$$\left\{ \begin{array}{ll} \dot{X}(t) + M\partial\phi(X(t)) \ni G(t, X(t)) & \text{a.e. on }]0, T[, \\ X(0) = \xi. & \end{array} \right.$$

Remark: $D(\partial\phi)$ is domain of $\partial\phi$ ($\partial\phi(z) \neq \emptyset$).

Numerical scheme

Let us consider A maximal monotone operator, f Lipschitz continuous and problem:

$$\begin{cases} \dot{u}(t) + A(u(t)) \ni f(t, u(t)) \\ u(0) = u_0. \end{cases}$$

♣ Numerical scheme: implicit Euler

♣ Convergence to the exact solution

♣ Non event-driven scheme

Let N be an integer. $h = T/N$. Let U^p solution of the numerical scheme

$$\begin{cases} \forall p \in \{0, \ldots, N-1\}, \frac{U^{p+1} - U^p}{h} + A(U^{p+1}) \ni f(ph, U^p) \\ U^0 = u_0 \end{cases}$$

Indeed:

$$\begin{cases} \forall p \in \{0, \ldots, N-1\}, U^{p+1} = (I + hA)^{-1}(hf(t_p, U^p) + U^p) \\ U^0 = u_0 \end{cases}$$

♣ From U^p to u_h linear approximation

♣ Convergence $h \longrightarrow 0$,

♣ **Proposition**

Order 1/2 for general maximal monotone graph. Under usual assumptions, there exists C such that for all h small enough,

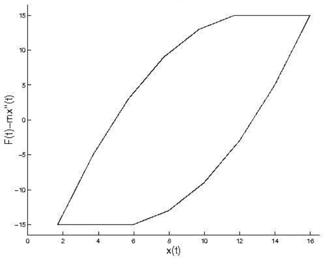
$$\left\{ \quad \forall t \in [0, T], \mid u(t) - u_h(t) \mid \leq C\sqrt{h}. \right.$$

Order 1 if A is sub-differential of potential. Under usual assumptions, there exists C such that for all h small enough,

$$\left\{ \quad \forall t \in [0, T], \mid u(t) - u_h(t) \mid \leq C \mid h \mid . \right.$$

Example of numerical results

♣ Under external solicitation e.g. periodic: Hysteresis cycles.

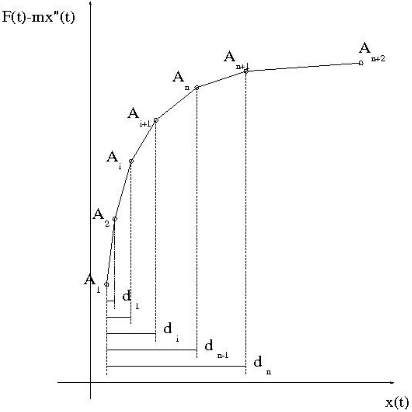


Example of generalized Prandtl model: $k_0 = 0$, $n = 5$, $k_i = 1$, $\eta_i = i$, $u_{0,i} = 0$, $F(t) = 10\cos(0.5t)$.

♣ From discrete data of half of loading curve of hysteresis cycles: identification.

Finding variations of smoothness (slopes) and abscissa of these changes permit identification.

Wavelet analysis to localize smoothness changes and to find slopes from discrete data..



Slopes p_j and abscissa d_j known...

$$\begin{cases} \forall j \in \{1,\ldots,5+1\}, \\ p_j = k_0 + \sum_{l=j}^{5} k_l, \\ \forall i \in \{1,\ldots,5\}, \\ d_i = 2\eta_i. \end{cases}$$

Other models

♣ Models with delay

$$\begin{cases} \dot{u}(t) + A(u(t)) + B(t, u(t)) + G(u(t-\tau)) \ni 0 \text{a.e. on }]0, T[, \\ \forall t \in [-\tau, 0], u(t) = z(t). \end{cases}$$

A maximal monotone operator, B, G Lipschitz continuous...

♣ Models with stochastic external solicitation

♣ Models with infinite number of S_i elements (continuous internal variable)

♣ Deterministic:

BASTIEN J., SCHATZMAN M., & LAMARQUE C.-H. Study of some rheological models with a finite number of degrees of freedom European Journal of Mechanics A/Solids, vol. 19, n 2, pp. 277-307, 2000.

BASTIEN J., SCHATZMAN M., & LAMARQUE C.-H. Study of an elastoplastic model with an infinite number of internal degrees of freedom European Journal of Mechanics. A, Solids, vol. 21, n 2, pp. 199-222, 2002.

♣ Deterministic + delay / memory:

LAMARQUE C.H., BASTIEN J., HOLLAND M.,Study of a Maximal Monotone Model with a Delay Term, SIAM Journal on Numerical Analysis, 41 (4), 2003, 1286-1300.

J. BASTIEN, C.-H. LAMARQUE, Maximal monotone model with history term, Nonlinear Analysis, Vol. 63, Issues 5-7, 30 November-15 December 2005, e199-e207

C.-H. LAMARQUE, J. BASTIEN, M. HOLLAND, Maximal monotone model with delay term of convolution, Mathematical Problems in Engineering, Vol. 2005, Issue 4, 437-453.

J. BASTIEN, C.-H. LAMARQUE, Non smooth dynamics of mechanical systems with history term, Nonlinear Dynamics, (2007), 47 : 115-128.

♣ Stochastic:

BERNARDIN F. Multivalued Stochastic Differential Equations: Convergence of a numerical scheme, Set-Valued Analysis, 11, 393-415, 2003.

F. BERNARDIN, M. SCHATZMAN, C.-H. LAMARQUE, Second-order multivalued stochastic differential equations on Riemannian manifolds, Proc. R. Soc. Lond. A (2004) 460, 1-28.

F. BERNARDIN, M. SCHATZMAN, C.-H. LAMARQUE, A stochastic differential equation from friction mechanics, C.R. Acad. Sci. Paris, Ser. I, 338, 837-842, 2004.

♣ Deterministic / Stochastic + locally Lipschitz continuous:

C.-H. LAMARQUE, F. BERNARDIN, J. BASTIEN, Study of a rheological model with friction term and cubic term : deterministic and stochastic case, European Journal of Mechanics A/Solids, 24 (2005) 572-592.

♣ Book:

AWREJCEWICZ J., & LAMARQUE C.-H. Bifurcation and chaos in nonsmooth mechanical systems. Vol. Series A. New Jersey, London, Singapore: World Scientific, 543 p., 2003

Summary

For every case, under assumptions and convenient theoretical frame

♣ **Existence and uniqueness**

♣ **Numerical case of Euler implicit type. Convergence**

♣ **Identification**

Remark 1: *Problems with uniqueness for "Coulomb friction" with time dependant friction coefficient...*

Remark 2: *Persoz pointed out other kinds of models. Questions: Is it possible to describe it using series or parallel configurations ? Same mathematical frame ? ...*

Schedule

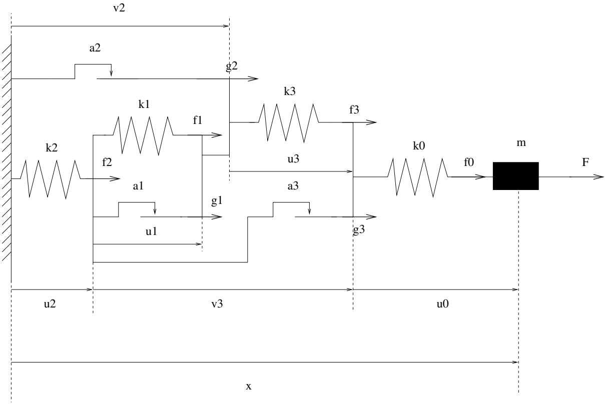
Reference:

B. Persoz, La rhéologie, Recueil de travaux des sessions de perfectionnement, INSA Lyon. Monographies du Centre d'Actualisation Scientifique et Technique, Masson, Paris, 1969. (In French).

Gephyroidal model: Persoz distinguishes "analyzable" models (i.e. models that can be splitted into branches settled either in series or in parallel) and gephyroidal model (similar to "bridge" $= \gamma\epsilon\phi\upsilon\rho\alpha$).

A simple example...

The rheological Persoz's gephyroidal model



We consider the model involving

1. four springs with stiffness k_0, k_1, k_2 and k_3
2. three St-Venant elements with threshold α_1, α_2 and α_3
3. one material point of mass m

Again:

The rheological Persoz's gephyroidal model, made out of some elementary rheological models (dry friction element and linear spring) can be covered by the existence and uniqueness theory for maximal monotone operators. Moreover, classical results of numerical analysis allow to use a numerical implicit Euler scheme, with order of convergence one. Some numerical simulations are presented.

Differential inclusion

The rheological Persoz's gephyroidal model is governed by differential inclusion of the form:

$$\begin{cases} \dot{X}(t) + MA(X(t)) \ni G(t, X(t)), \text{a.e. on }]0, T[, \\ X(0) = X_0, \end{cases}$$

where

M is a invertible matrix

X is a function from $[0, T]$ in $\mathbb{R}^p$

A is a maximal monotone graph on $\mathbb{R}^p$

G a function from $[0, T] \times \mathbb{R}^p$ in $\mathbb{R}^p$

Notations

We introduce the classical following notations :

- for $i \in \{0, ..., 3\}$, $k_i \longrightarrow$
  - displacements u_i
  - forces f_i
- for $i \in \{2, 3\}$, $\alpha_i \longrightarrow$
  - displacements v_i
  - forces g_i
- k_1 and $\alpha_1 \longrightarrow$
  - displacements $v_1 = u_1$
  - forces g_1
- Let x be the abscissa of material point with mass m, and F be the external force, applied to this point.

Matrices in the model are

$$\begin{cases} M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & K \end{pmatrix}, X = (x, \dot{x}, g_2, g_3, g_1)^T. \\ K = \begin{pmatrix} k_0 + k_2 & k_0 & -(k_0 + k_2) \\ k_0 & k_0 + k_3 & -(k_0 + k_3) \\ -(k_0 + k_2) & -(k_0 + k_3) & k_0 + k_1 + k_2 + k_3 \end{pmatrix} \end{cases}$$

after eliminating all the other unknowns. Let us define convex set :

$$\mathcal{C} = \mathbb{R} \times \mathbb{R} \times [-\alpha_2, \alpha_2] \times [-\alpha_3, \alpha_3] \times [-\alpha_1, \alpha_1]$$

The differential inclusion governing the model

After computation, we obtain

$$\dot{X}(t) + M\partial\psi_{\mathcal{C}}(X(t)) \ni G\big(t, X(t)\big),$$

where

- X is a function from $[0, T]$ in $\mathbb{R}^5$
- M is a symmetric positive definite matrix (under some assumptions)
- $\partial\psi_{\mathcal{C}}$ is the subdifferential of the indicatrix of a closed convex of $\mathbb{R}^5$ for the scalar product defined by

$$< X, Y >_M = X^T M^{-1} Y,$$

- G is a regular function from $[0, T] \times \mathbb{R}^5$ in $\mathbb{R}^5$

Existence and uniqueness

Theorem (Existence and uniqueness)

Let $(\alpha_i)_{1\le i\le 3}$ *be positive numbers,* $(k_i)_{0\le i\le 3}$ *be positive numbers satisfying*

1. $k_0 = 0$ *and for all* $i \in \{1,2,3\}$, $k_i > 0$
2. *or* $k_0 > 0$ *and at least two numbers among* k_1, k_2 *and* k_3 *are non negative.*

There is a unique solution X *in* $W^{1,\infty}(0,T;\mathbb{R}^5)$ *for the previous differential inclusion.*

Main idea of the proof ...

... is based on the following idea : if $\mathbb{R}^5$ is equipped with its canonical scalar product, and with another scalar product

$$< X, Y >_M = X^T M^{-1} Y,$$

where M is symmetric positive definite, then we can relate the sub-differential $\partial\phi$ of ϕ relatively to the canonical scalar product and the sub-differential $\partial_M \phi$ relatively to $<, >_M$ by

$$\partial_M \phi(X) = M \partial\phi(X).$$

We apply then results proved in :

J. Bastien and M. Schatzman, *Numerical precision for differential inclusions with uniqueness*, M2AN. Mathematical Modelling and Numerical Analysis, 36 (3), 2002, 427–460.

J. Bastien and M. Schatzman, *Schéma numérique pour des inclusions différentielles avec terme maximal monotone*, Comptes Rendus de l'Académie des Sciences. Série I. Mathématique, 330 (7), 2000, 611–615.

Numerical scheme

Theorem

Let N be an integer, $h = T/N$, $h_p = hp$ and X^p defined by

$$X^{p+1} = proj_{\mathcal{C},M^{-1}}\left(X^p + G(t_p, X^p)\right),$$

where $proj_{\mathcal{C},M^{-1}}$ is the orthogonal projection on the convex $\mathcal{C}$ for the previously defined norm on $\mathbb{R}^5$. Denote $X_h \in C^0\left([0,T];\mathbb{R}^5\right)$ the linear interpolation at time $t_p = hp$ of the solution X^p.

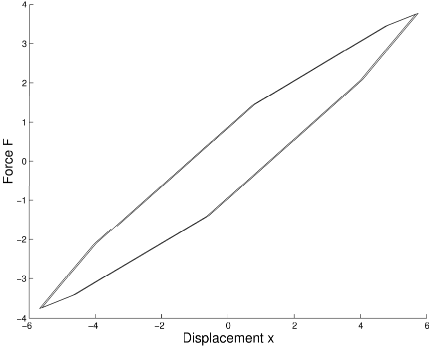
The numerical scheme is of first order.

Quasistatic problems

In the quasistatic case, the mass m can be equal to zero.
Existence, uniqueness and numerical scheme hold.

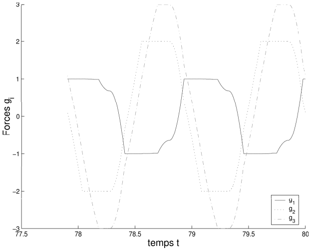
Numerical simulations : hysteresis cycle

After transient: cycle (hysteresis)



Numerical results : Specificity of gephyroidal model

Opposite variations of 2 g_i versus the last one.



Schedule

Conclusions

♣ Modelling with non smooth simple Saint-Venant elements: Another case Persoz's model

♣ Non classical one

♣ Mathematically and numerically similar to various cases previously examined

♣ Outlook:

- Generalization to "bridges" assembling.
- Applications and identification.