A CHARACTERISATION OF THE BOUNDARIES OF PLANAR WORKSPACES

JÉRÔME BASTIEN*, PIERRE LEGRENEUR**, AND KARINE MONTEIL†

ABSTRACT. A planar polyarticulated system was modelised by points defining the joints and a last point A_p linked to the last solid. The surface swept by the point A_p has its boundary defined by 3 kinds of particular configurations. These boundaries are classically characterised in the litterature by symbolically vanishing some determinants of jacobian of position functions. However, this method requires the resolution of important sets of non linear equations.

So, the main purpose of this paper is to propose a pure geometrical resolution of the problem in a planar case, in order to avoid these computations. In this aim, a simply geometrical interpretation of jacobian's singularities is used. This new formulation is applied to a human free moving arm for describing the workspace of its distal extremity (i.e. the finger).

The relevance of this work in sport locomotion is important. Indeed, it allows to predict the degrees of freedom recruted for a given mouvement as a function of the position of the target in a given workspace.

1. INTRODUCTION

This work correspond to a short version of submitted or accepted previous works [BLM06, BLM07].

In the aim to reach a target or realize an explosive movement like jumping, the degrees of freedom (Dof) of the articular chain allow the recruitment of an infinity combination of these ones. Each combination of Dof define a specific workspace. Thus, the knowledge of the position of the target to reach could allow to predict the articulations engaged in the movement. For example, the displacement of the distal extremity of an articular chain of segments, like a human arm, result from the transformation of rotational kinetic energy of the involved joints in linear kinetic energy of the extremity (i.e. the finger). The space defined by this extremity for a finite number of degree of freedom is so-called «workspace» of the finger. The boundary to the workspace is called the «reach envelop» [Mol98].

Some works study the boundaries [HS05, HS02, HS00, CZM06, DM99, ECB06, MGM98], however, as far as we know, no analytical solution of this description as well as automatic method for describing the boundaries are available in the literature.

In dimension $n \in \{2, 3\}$, the workspace is considered as the range of a convex polytope of $\mathbb{R}^p$ ($n \leq p$) by a differentiable function Φ_p. Here p is the number of independant parameters. In biomechanics or robotics, one of most known method for determining this boundary is to write that jacobian of function Φ_p is necessary singular on the boundaries [AMAYH97, AMY97, AMYS98, AMYZT04, DPH01]. To obtain the different barriers defined by this singular jacobian of Φ_p, equations are given by vanishing all determinants of jacobian. This computation are done through symbolic calculation. Simplifications allow to improve the algorithms

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∗ Laboratoire Mécatronique 3M, Équipe d'accueil A 3318, Université de Technologie de Belfort-Montbéliard, 90010 Belfort cedex, France. email adress: `jerome.bastien@utbm.fr`.

∗∗ Laboratoire d'Analyse de la Motricité et Apprentissage (LAMA), Centre de Recherche et d'Innovation sur le Sport (CRIS), Equipe d'Accueil EA 647, Université Claude Bernard Lyon 1, France. email adress: `pierre.legreneur@univ-lyon1.fr`.

† (LAMA), email adress: `karine.monteil@univ-lyon1.fr`.

without resolving the problem of symbolic calculation. Indeed this method is very expensive in calculation time, espacially when the number of degrees of freedom increase.

So, the main purpose of this paper is to propose an algorithm for describing the surface as joining arcs of circles, determinable through simple calculation rules. In this aim, a simple geometrical interpretation of jacobian's singularities will be used for $n = 2$.

2. Theoritical basis and presentation of the studied problem

Let $\left(O, \vec{i}, \vec{j}\right)$ be a reference frame, p an integer greater than or equal to 2, $(l_i)_{1\leq i\leq p}$ a p non negative numbers and $\left(\theta_i^+\right)_{1\leq i\leq p}$ and $\left(\theta_i^-\right)_{1\leq i\leq p}$ $2p$ angles satisfying

$$\forall i \in \{1, ..., p\}, \quad -\pi < \theta_i^- < \theta_i^+ \leq \pi. \tag{1}$$

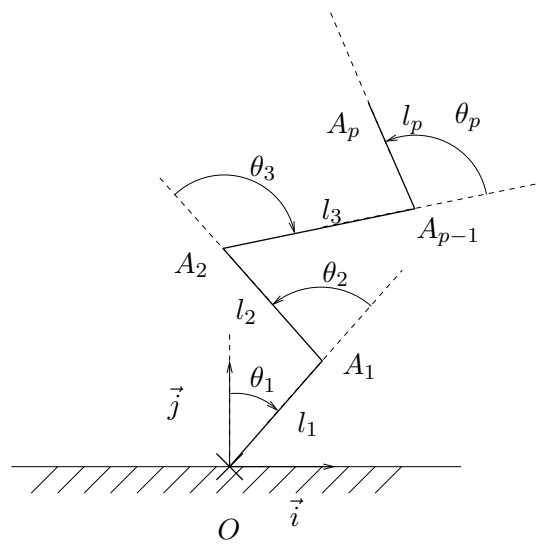


FIGURE 1. The considered planar system.

We define the workspace as the set of points A_p such as (see Fig. 1)

$$A_0 = 0, \quad \widehat{\left(\vec{j}, \overrightarrow{0A_1}\right)} = \theta_1, \tag{2a}$$

$$\forall i \in \{2, ..., p\}, \quad \widehat{\left(\overrightarrow{A_{i-2}A_{i-1}}, \overrightarrow{A_{i-1}A_i}\right)} = \theta_i, \quad \forall i \in \{1, ..., p\}, \quad A_{i-1}A_i = l_i, \tag{2b}$$

with the constraints

$$\forall i \in \{1, ..., p\}, \quad \theta_i \in [\theta_i^-, \theta_i^+]. \tag{2c}$$

We consider function Φ_p from domain $F = \prod_{i=1}^{p} [\theta_i^-, \theta_i^+]$ to $\mathbb{R}^2$ defined by

$$\forall (\theta_1, ..., \theta_p) \in F, \quad \Phi_p\big(\theta_1, ..., \theta_p\big) = A_p. \tag{3}$$

All the elements $x = (\theta_1, ..., \theta_p)$ of domain F satisfy (2c). Thus, according to constrained optimization technics, consider the following definition :

Definition 1. For all $x = (\theta_1, ..., \theta_p) \in F$, for all $i \in \{1, ..., p\}$, the i-constraint (2c) is active if $\theta_i \in \{\theta_i^-, \theta_i^+\}$ and inactive if $\theta_i \in]\theta_i^-, \theta_i^+[$, which means that $\theta_i \in \{\theta_i^-, \theta_i^+\}$ is saturated and $\theta_i \in]\theta_i^-, \theta_i^+[$ is free.

We try to determine the topological boundary $\partial D = \overline{D} \setminus \overset{\circ}{D}$ of $D = \Phi_p(F)$. Since F is continuous and F is compact, D is compact and $\partial D = D \setminus \overset{\circ}{D}$.

We refer to Appendix A, where some recalls about the jacobian of Φ_p on the boundary of F are given. Thus, results of the previous appendix will be applied with F, $\Phi = \Phi_p$, $p \geq 2$ and $n = 2$.

3. Behaviour of jacobian of application Φ_p on the boundary of workspace

Refering to Definition 1, we will present a consequence of this lemma fundamental for the geometrical interpretation of results of Appendix A.

Let $x = (\theta_1, ..., \theta_p)$ and $y = \Phi_p(x)$ such that the number q of free components of x belongs to $\{2, ...p\}$. Denote $I = \{i_1, ..., i_q\}$ the set of integers $1 \leq i_1 < i_2 < ... < i_q \leq p$ corresponding to free components of x and $J = \{j_1, ..., j_{p-q}\}$ the set of integers $1 \leq j_1 < j_2 < ... < j_{p-q} \leq p$ corresponding to saturated components of x. The sets I and J define a partition of $\{1, ..., p\}$ and we have, by using Proposition 7 with $n = 2$ and $\Phi = \Phi_p$

Proposition 2. *The element $y = \Phi_p(\theta_1, ..., \theta_p)$ belongs to $S_I \cup S_{II}$ if and only if*

$$\text{the } q+1 \text{ points } A_{i_1-1}, A_{i_2-1}, ..., A_{i_q-1} \text{ and } A_p \text{ are aligned.} \tag{4}$$

Remark 1. In all this paper, it is assumed that, for all pair of integers (i, j), if $i \neq j$, then A_i and A_j are distinct, which holds in biomechanics and robotics.

Remark 2. The geometrical of idea of Proposition 2 is very simple. It is proposed for three aligned points in [MGM98], but it is not generalized. It permits to write the algorithm of description of $S = S_{\mathrm{I}} \cup S_{\mathrm{II}} \cup S_{\mathrm{III}}$, presented in Section 4 and in [BLM06, BLM07].

Thanks to proposition 2, we can prove that that point y belongs to $S_{\mathrm{I}} \cup S_{\mathrm{II}}$ if and only if:

- each of saturated components θ_{j_k} for $1 \leq k \leq p - q$ is known;
- and each of free components θ_{i_k} for $2 \leq k \leq q$ is known according to the previous saturated components;
- and only the free component θ_{i_1} describes the interval $]\theta_{i_1}^-, \theta_{i_1}^+[$.

4. Geometrical definition of $S = S_{\mathrm{I}} \cup S_{\mathrm{II}} \cup S_{\mathrm{III}}$ as finite union of arcs of circles

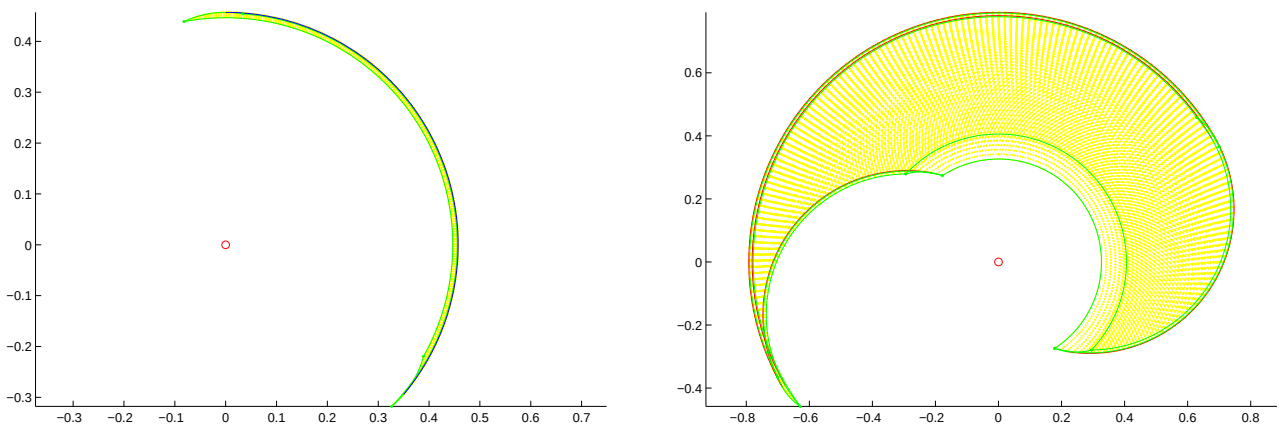
Proposition 3. *The part S_{III} is a finite union of arcs of circles and each of them is defined by $\Phi_p\big(\theta_1, ..., \theta_{i-1}, [\theta_i^-, \theta_i^+], \theta_{i+1}, ..., \theta_p\big)$ where i describes $\{1, ..., p\}$ and*

$$\forall j \neq i, \quad \theta_j \in \{\theta_j^-, \theta_j^+\}. \tag{5}$$

The new and original results of the paper are the two following:

Proposition 4. *If for all $j \in \{2, ..., p\}$, $\theta_j^- \theta_j^+ < 0$ then S_I is the arc of circle defined by $\Phi_p\big(]\theta_1^-, \theta_1^+[, 0, 0, ..., 0\big)$, else S_I is empty.*

In the second case, S_{I} corresponds to the maximal extension of the arm. The point A_p describes a circle of radius $\sum_{i=1}^{p} l_i$, which is the greatest possible distance to the origin .



(a) (b)

FIGURE 2. Simulations corresponding to cases 1 (a) and 2 (b); the discret swept volume is ploted in yellow, S_{I} is ploted in blue, S_{II} is ploted in red and S_{III} is ploted in green.

Proposition 5. *There exist an integer $M \in \mathbb{N}$, m integers $(p_m)_{1\leq m\leq M}$ of $\{1,...,p\}^M$, M elements of $\mathbb{R}^{p-1}$, $\left(\theta_1^m,...,\theta_{p_m-1}^m,\theta_{p_m+1}^m,...,\theta_p^m\right)_{1\leq m\leq M}$ and $2m$ numbers $\{\theta_m^-,\theta_m^+\}_{1\leq m\leq M}$ (with $\theta_m^- < \theta_m^+$) such that S_{II} is the finite union of arcs of circles defined by*

$$\bigcup_{1\leq m\leq M} \Phi_p\left(\theta_1^m,...,\theta_{p_m-1}^m,]\theta_m^-,\theta_m^+[,\theta_{p_m+1}^m,...,\theta_p^m\right). \tag{6}$$

5. NUMERICAL SIMULATIONS

Now will be presented some numerical simulations with the shape of $S = S_{\mathrm{I}} \cup S_{\mathrm{II}} \cup S_{\mathrm{III}}$.

cases	p	Segment length / body height	$\left(\theta_i^-\right)_{1<i<p}$ (°)	$\left(\theta_i^+\right)_{1<i<p}$ (°)	figures
1	2	0.146, 0.108	-130, -10	0, 25	2(a)
2	3	0.186, 0.146 , 0.108	-60, -130, -10	120, 0, 25	2(b)

TABLE 1. For simulation, one subject of 1.80 m height is considered. Lengths of the upper limb were determided from anthropometric data [Win90]. Thus, segment lengths are presented as percent of total body height (0.108, 0.146 and 0.186 for the hand, forearm and upperarm respectively). Angles correspond to minima and maxima of joints degrees of freedom of human upper limb, i.e. shoulder abduction/adduction ($-60°/120°$), elbow flexion/extension ($-130°/0°$), wrist abduction/adduction ($-10°/25°$).

(1) Case 1 corresponds to forearm and hand displacements;
(2) Case 2 corresponds to upperarm, forearm and hand displacements.

Each figure presents the computed arcs of circles, and, for $p \geq 2$, the discret swept volume obtained by ploting the set $\Phi_p(\theta_1^m,...,\theta_p^m)$ where $(\theta_1^m,...,\theta_p^m)$ belongs to a finite set of $[\theta_1^-,\theta_1^+]\times[\theta_2^-,\theta_2^+]\times...\times[\theta_p^-,\theta_p^+]$.

On Figures 2, the part S_{I} is included on the boudary of the workpace, which is not the case for S_{II} and S_{III}. Indeed, the geometrical condition for describing the boundary is necessary but non sufficient. Thus, some

arcs of circles have to be remove, aimed to describe only the boundary of the workspace. Theses difficulties are also pointed out by [AMAYH97, AMY97, AMYS98].

6. Conclusion

The part of the boundary is traditionally written under the form $S = S_{\mathrm{I}} \cup S_{\mathrm{II}} \cup S_{\mathrm{III}}$. The part S_{II} corresponds to points for which only one of constraint is inactive. The parts S_{I} and S_{III} corresponds to points for which at least two constraints are inactive.

This paper shows that the resolution of the problem under jacobian formulation is not necessary. Indeed, considering that all the points for which the constraints are inactive are aligned suffied to describe the workpace boundaries.

Moreover, this geometrical formulation condition gives a phenomenological description of the boundary, which locally corresponds to a position for which the considered joint is partially extended.

Appendix A. Recalls about fundamental theoretical results

The aim this appendix is to remain some theoretical fundamental lemmas, where it can find presentation for example in [AMYZT04, AMAYH97, AMY97].

Let p and n be two integers satisfying $p \geq n \geq 1$ and Φ a function from $\mathbb{R}^p$ to $\mathbb{R}^n$, whose domain is a compact set F. Consider $D = \Phi(F)$ and $\partial D = \overline{D} \setminus \overset{\circ}{D}$ the boundary of D.

Assuming that Φ is of class C^1 on F and that F is the convex polytope of $\mathbb{R}^p$ defined by

$$F = \prod_{i=1}^{p} [\alpha_i, \beta_i], \tag{7}$$

where $\alpha_i < \beta_i$.

The differential and the jacobian matrix of Φ at x are identified i. e.:

$$\forall (i,j) \in \{1, ..., n\} \times \{1, ..., p\}, \quad (d\Phi(x))_{i,j} = \frac{\partial \phi_i}{\partial x_j}(x). \tag{8}$$

The fundamental following results are:

Lemma 6. *Let x be an element of F such that $\Phi(x)$ belongs to ∂D. Let $q \in \{0, ..., p\}$ the number of free components of x. There are three exclusive cases:*

(1) *If $q = p$, then*

$$\textit{rank}\ \big(d\Phi(x)\big) \leq n - 1. \tag{9}$$

(2) *If $n \leq q \leq p - 1$, denote by $\widetilde{d\Phi(x)}$ the submatrix of $d\Phi(x)$, where all the columns corresponding to the saturated components of x are removed. Then*

$$\textit{rank}\ \left(\widetilde{d\Phi(x)}\right) \leq n - 1. \tag{10}$$

(3) *If $q \leq n - 1$, there is no condition on the jacobian.*

Proposition 7. *The three surfaces S_I, S_{II} and S_{III} of $\mathbb{R}^n$, corresponding to the three exclusive cases of Lemma 6, are defined by:*

$$\Phi(x) \in S_I \Longleftrightarrow q = p, \tag{11a}$$

$$\Phi(x) \in S_{II} \Longleftrightarrow q \in \{n, ..., p-1\}, \tag{11b}$$

$$\Phi(x) \in S_{III} \Longleftrightarrow q \leq n - 1. \tag{11c}$$

Then, the boundary ∂D of D is included in $S = S_I \cup S_{II} \cup S_{III}$.

These results give a necessary but non sufficient condition for being on the boundary of D.

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