Boundaries of the polyarticulated system workspace in the plane

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Abstract

We model a planar polyarticulated system by points defining the joints and a last point A_p linked to the last solid. The surface swept by the point A_p has its boundary defined by 3 kinds of particular configurations. These curves can be geometrically determinated. These results come out from two papers [BLM07, BLM06].

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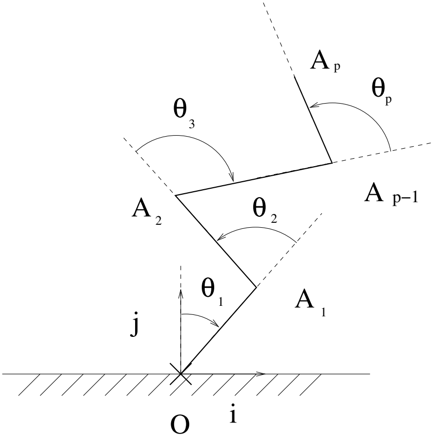
The plan considered system



We assume that

$$\forall i \in \{1, ..., p\}, \quad -\pi < \theta_i^- < \theta_i^+ \leq \pi. \tag{1}$$

We define the workspace as the set of points A_p such as

$$A_0 = 0, \tag{2a}$$

$$\widehat{\left(\vec{j}, \overrightarrow{0A_1}\right)} = \theta_1, \tag{2b}$$

$$\forall i \in \{2, ..., p\}, \quad \widehat{\left(\overrightarrow{A_{i-2}A_{i-1}}, \overrightarrow{A_{i-1}A_i}\right)} = \theta_i, \tag{2c}$$

$$\forall i \in \{1, ..., p\}, \quad A_{i-1}A_i = l_i, \tag{2d}$$

with the constraints

$$\forall i \in \{1, ..., p\}, \quad \theta_i \in [\theta_i^-, \theta_i^+]. \tag{2e}$$

We consider function Φ_p from domain

$$F = \prod_{i=1}^{p} [\theta_i^-, \theta_i^+], \tag{3}$$

to $\mathbb{R}^2$ defined by

$$\forall (\theta_1, \dots, \theta_p) \in F, \quad \Phi_p\big(\theta_1, \dots, \theta_p\big) = A_p. \tag{4}$$

Definition

For all $x=(\theta_1,....,\theta_p)\in F$, for all $i\in\{1,...,p\}$, the constraint $\theta_i\in[\theta_i^-,\theta_i^+]$ is

- active if $\theta_i\in\{\theta_i^-,\theta_i^+\}$
- inactive if $\theta_i\in]\theta_i^-,\theta_i^+[$,

which means that

- $\theta_i\in\{\theta_i^-,\theta_i^+\}$ is saturated
- $\theta_i\in]\theta_i^-,\theta_i^+[$ is free.

Aim

We try to determine the topological boundary $\partial D = \overline{D} \setminus \overset{\circ}{D} = \overline{D} \setminus D$ of $D = \Phi_p(F)$ where F is the convex polytope of $\mathbb{R}^p$ defined by (3).

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See [AMYZT04, AMYS98, AMAYH97, AMY97].
Let

- p and n such that $p \geq n \geq 1$;
- Φ a function from $\mathbb{R}^p$ to $\mathbb{R}^n$, with compact set domain given by

$$F = \prod_{i=1}^{p} [\alpha_i, \beta_i], \tag{5}$$

- $D = \Phi(F)$

Lemma

Let x be an element of F such that $\Phi(x)$ belongs to ∂D. Let $q \in \{0, ..., p\}$ the number of free components of x. There are three exclusive cases:

1. *If $q = p$, then rank $\left(d\Phi(x)\right) \leq n - 1$.*
2. *If $n \leq q \leq p - 1$, denote by $\widetilde{d\Phi(x)}$ the submatrix of $d\Phi(x)$, where all the columns corresponding to the saturated components of x are removed. Then rank $\left(\widetilde{d\Phi(x)}\right) \leq n - 1$.*
3. *If $q \leq n - 1$, there is no condition on the jacobian.*

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Theorem

The three surfaces S_I, S_{II} and S_{III} of $\mathbb{R}^n$, corresponding to the three exclusive cases of previous Lemma, are defined by:

$$y \in S_I \Longleftrightarrow \exists x, \quad y = \Phi(x) \textit{ and } q = p, \tag{6a}$$

$$y \in S_{II} \Longleftrightarrow \exists x, \quad y = \Phi(x) \textit{ and } q \in \{n, ..., p-1\}, \tag{6b}$$

$$y \in S_{III} \Longleftrightarrow \exists x, \quad y = \Phi(x) \textit{ and } q \leq n-1. \tag{6c}$$

Then, the boundary ∂D of D is included in $S = S_I \cup S_{II} \cup S_{III}$.

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A geometrical lemma on the jacobian of Φ_p

Lemma

Let $k \in \{1,\dots,p\}$, $(\theta_j)_{j\in\{1,\dots,p\}\setminus\{k\}}$, be $p-1$ fixed angles and $\widetilde{\Phi}$ a function from $\mathbb{R}$ to $\mathbb{R}^2$ defined by

$$\widetilde{\Phi}(\theta) = \Phi_p\left(\theta_1,\dots,\theta_{k-1},\theta,\theta_{k+1},\dots,\theta_p\right). \tag{7}$$

Then, the range of $d\widetilde{\Phi}(\theta_k)$ is a line orthogonal to $(A_{k-1}(\theta_1,\dots,\theta_{k-1}), A_p(\theta_1,\dots,\theta_p))$.

Proof.

If θ is varying, point $A_p = \Phi_p(\theta_1,\ \dots,\ \theta_{k-1},\theta,\ \theta_{k+1},\dots,\theta_p)$ describes a circle of center A_{k-1}. □

Idea allready seen (but non used) in [MGM98].

- Let $y = \Phi_p(x)$ be an element of $\mathbb{R}^2$ such that the number q of free components of x belongs to $\{2, ...p\}$.
- Denote
  - $I = \{i_1, ..., i_q\}$ the set of integers $1 \leq i_1 < i_2 < ... < i_q \leq p$ corresponding to free components of x
  - $J = \{j_1, ..., j_{p-q}\}$ the set of integers $1 \leq j_1 < j_2 < ... < j_{p-q} \leq p$ corresponding to saturated components of x.
- The sets I and J define a partition of $\{1, ..., p\}$ and we have :

The main (simple) result

Theorem

The element $y = \Phi_p(\theta_1, \ldots, \theta_p)$ belongs to $S_I \cup S_{II}$ if and only if

$$\text{the } q+1 \text{ points } A_{i_1-1},\ A_{i_2-1},\ \ldots,\ A_{i_q-1} \text{ and } A_p \text{ are aligned.} \tag{8}$$

Proof.

- Since p is greater than 2, the rank of submatrix of jacobian matrix $\widetilde{d\Phi_p}(x)$ is equal to one.
- The previous Lemma could then be applied in the case where only one of free components of θ_{i_k} of x among θ_{i_1}, θ_{i_2}, ..., θ_{i_q} is varying: for all $k \in \{1, \ldots, q\}$, the range of $\widetilde{d\Phi_p}(x)$ is a line orthogonal to (A_{i_k-1}, A_p).
- The reciprocal is identical.

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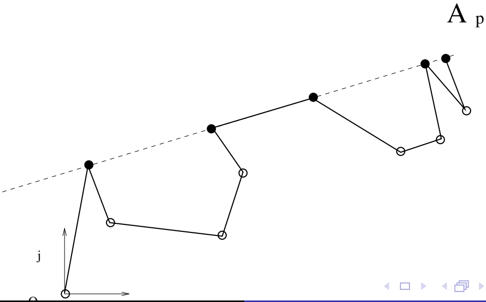
$\square$

Analytical consequence of the main result

- Point A_i where the constraint is inactive (free angle, ploted by $\bullet$);
- Point A_i where the constraint is active (saturated angle, ploted by $\circ$).



Analytical consequence of the main result

We can then deduce that that point $y = \phi_p(x)$ belongs to $S_{\mathrm{I}} \cup S_{\mathrm{II}}$ if and only if:

- each of saturated components θ_{j_k} for $1 \leq k \leq p - q$ is known;
- each of free components θ_{i_k} for $2 \leq k \leq q$ is known according to the previous saturated components;
- only the free component θ_{i_1} describes the interval $]\theta_{i_1}^-, \theta_{i_1}^+[$.

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Geometrical and algorithmic definition

Theorem

The part S_{III} is a finite union of arcs of circles and each of them is defined by $\Phi_p(\theta_1, \ldots, \theta_{i-1}, [\theta_i^-, \theta_i^+], \theta_{i+1}, \ldots, \theta_p)$ where i describes $\{1, \ldots, p\}$ and

$$\forall j \neq i, \quad \theta_j \in \{\theta_j^-, \theta_j^+\}.$$

Theorem

If for all $j \in \{2, \ldots, p\}$, $\theta_j^- \theta_j^+ < 0$ then S_I is the arc of circle defined by $\Phi_p(]\theta_1^-, \theta_1^+[, 0, 0, \ldots., 0)$, else S_I is empty.

Geometrical and algorithmic definition

Theorem

There exist

- *an integer M,*
- *m integers $(p_m)_{1\leq m\leq M}$,*
- *M elements of $\mathbb{R}^{p-1}$, $\left(\theta_1^m, \dots, \theta_{p_m-1}^m, \theta_{p_m+1}^m, \dots, \theta_p^m\right)_{1\leq m\leq M}$*
- *$2m$ numbers $\{\theta_m^-, \theta_m^+\}_{1\leq m\leq M}$ (with $\theta_m^- < \theta_m^+$)*

such that S_{II} is the finite union of arcs of circles defined by

$$\bigcup_{1\leq m\leq M} \Phi_p\left(\theta_1^m, \dots, \theta_{p_m-1}^m,]\theta_m^-, \theta_m^+[, \theta_{p_m+1}^m, \dots, \theta_p^m\right).$$

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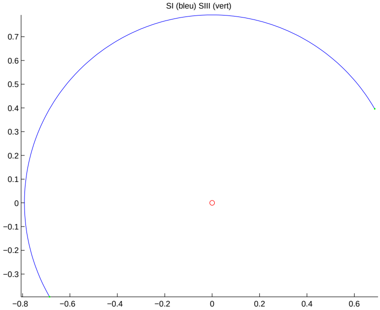
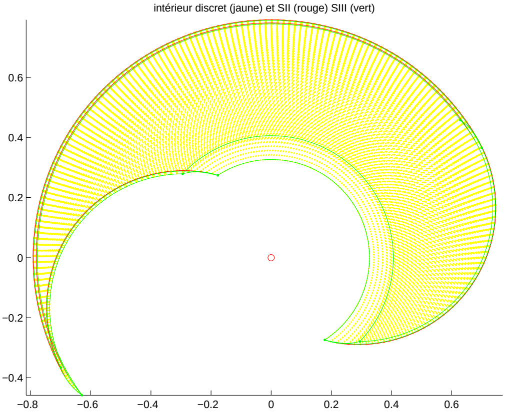
Now will be presented some numerical simulations with the shape of $S = S_{\mathrm{I}} \cup S_{\mathrm{II}} \cup S_{\mathrm{III}}$ for the arm of a subject of 1.80 m height, with 1, 2 or 3 degrees of freedom.

1 or 2 dof results



3 dof results



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Automatically and better algorithm

- Pure geometrically description of boundary;
- Improvement of existing algorithms, based on symbolic calculation [AMYZT04, AMYS98, AMAYH97, AMY97, DPH01]

Improvements

- Give sufficient conditions;
- Extend in $\mathbb{R}^3$, by using angles and matrix of Denavit-Hartenberg [DH55].

Applications

Associated with works on inverse geometry and time joint description (in process), this study permits to modelize and simulate the sporty gesture, like dynamical jump or locomotor pointing.

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